Hamiltonian Dynamical Systems and Galactic Dynamics

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2 Participants

2.1 ICRANet participants

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3 Brief description

3.1 Near-integrable dynamics and galactic structures.

The study of self-gravitating stellar systems has provided in several occasions important hints to develop powerful tools of analytical mechanics. We may cite the ideas of Jeans (1929) about the relevance of conserved quantities in describing the phase-space structure of large N-body systems and his introduction of the concept of *isolating integral*. Later important contribution are those of Hénon & Heiles (1964), where a paradigmatic example of non-integrable system derived from a simple galactic model was introduced and of Hori (1966), where the theory of Lie transform was introduced in the field of canonical perturbation theory and Hamiltonian normal forms. These and other cues contributed to set up a body of methods and techniques to analyze the near integrable and chaotic regimes of the dynamics of generic non-integrable systems. For a general overview see Boccaletti & Pucacco (Theory of Orbits, Vol. I, 1996; Vol. II, 1999).

On the other side, the payback from analytical mechanics to galactic dynamics has not been as systematic and productive as it could be. The main line of research has been that pursued by Contopoulos (2002) who applied a direct approach to compute approximate forms of effective integrals of motion. The method of Hori (1966), subsequently developed by several other people (Deprit, 1969; Dragt & Finn, 1976; Efthymiopoulos et al. 2004; Finn, 1984; Giorgilli, 2002), has several technical advantages and has gradually become a standard tool in the perturbation theory of Hamiltonian dynamical systems (Boccaletti & Pucacco, 1999). In this respect we have applied the Lie transform method to construct Hamiltonian normal forms of perturbed oscillators and investigate the orbit structure of potentials of interest in galactic dynamics (Belmonte, Boccaletti and Pucacco, 2006, 2007a, 2007b, 2008; Pucacco et al. 2008a, 2008b). The approach allows us to gather several informations concerning the near integrable dynamics below the stochasticity threshold (if any) of the system. Being a completely analytical approach, it has the fundamental value of a complete generality which provides simple recipes to explore the structure of the backbone of phase-space. Exploiting asymptotic properties of the series constructed via the normal form, one can also get quantitative predictions extending the validity of the approach well beyond the radius of convergence of the initial series expansion of the perturbed potential. We have shown how to exploit resonant normal forms to extract information on several aspects of the dynamics of the the logarithmic and the Schwarzschild potential. In particular, using energy and ellipticity as parameters, we have computed the instability thresholds of axial orbits, bifurcation values of low-order boxlets and phase-space fractions pertaining to the families around them. We have also shown how to infer something about the singular limit of the potential.

As in any analytical approach, this method has the virtue of embodying in (more or less) compact formulas simple rules to compute specific properties, giving a general overview of the behavior of the system. In the case in which a non-integrable system has a regular behavior in a large portion of its phase space, a very conservative strategy like the one adopted in our work provides sufficient qualitative and quantitative agreement with other more accurate but less general approaches. In our view, the most relevant limitation of this approach, common to all perturbation methods, comes from the intrinsic structure of the single-resonance normal form. The usual feeling about the problems posed by non-integrable dynamics is in general grounded on trying to cope with the interaction of (several) resonances. Each normal form is instead able to correctly describe only one resonance at the time. However, we remark that the regular dynamics of a non-integrable system can be imagined as a superposition of very weakly interacting resonances. If we are not interested in the thin stochastic layers in the regular regime, each portion of phase space associated with a given resonance has a fairly good alias in the corresponding normal form. An important subject of investigation would therefore be that of including weak interactions in a sort of higher order perturbation theory. For the time being, there are two natural lines of developement of this work: 1. to extend the analysis to cuspy potentials and/or central 'black holes'; 2. to apply this normalization algorithm to three degrees of freedom systems.

Our most recent results concern the investigation of bifurcations of families of periodic orbits ('thin' tubes) in axisymmetric galactic potentials (Pucacco, 2009). We verify that the most relevant bifurcations are due to the (1:1) resonance producing the 'inclined' orbits through two different mechanisms: from the disk orbit and from the 'thin' tube associated to the vertical oscillation. The closest resonances occurring after these are the (4:3) resonance in the oblate case and the (2:1) resonance in the prolate case. The (1:1) resonances are treated in a straightforward way using a 2nd-order truncated normal form. The higher-order resonances are instead cumbersome to investigate, because the normal form has to be truncated to a high degree and the number of terms grows very rapidly. We therefore adopt a further simplification giving analytic formulas for the values of the parameters at which bifurcations ensue and compare them with selected numerical results. Thanks to the asymptotic nature of the series involved, the predictions are reliable well beyond the convergence radius of the original series.

3.2 Geometric approach to the integrability of Hamiltonian systems

Integrable systems are still very useful benchmarks to understand the properties of general non-integrable systems, not only for their relevance as starting points for perturbation theory. The topology of invariant surfaces in the phase-space of integrable systems can be highly non-trivial and give rise to complex phenomena (high-order resonances, monodromy, etc.) still not completely understood.

We have started our work by investigating quadratic integrals at fixed and arbitrary energy with a unified geometric approach (Rosquist & Pucacco, 1995; Boccaletti & Pucacco, 1997) solving the Killing tensor equations for 2nd-rank Killing tensors on 2-dim. conformally-Euclidean spaces. In Pucacco & Rosquist (2003) these systems have been shown to be endowed of a bi-Hamiltonian structure and in Pucacco & Rosquist (2004) a class of systems separable at fixed energy has been shown to be non-integrable in Poincarè sense. The case of cubic and quartic integrals of motion, respectively associated to 3rd and 4th-rank Killing tensors, has been investigated in Karlovini & Rosquist (2000) and in Karlovini, Pucacco, Rosquist and Samuelsson (2002). In Pucacco (2004) and Pucacco & Rosquist (2005a) we have obtained new classes of integrable Hamiltonian system with vector potentials and in Pucacco & Rosquist (2005b) we have provided a general treatment of weakly integrable systems.

Recently, Pucacco & Rosquist (2007, 2009), we have presented the theory of separability over 2-dimensional pseudo-Riemannian manifolds ("1+1" separable metrics) by classifying all possible separability structures and providing some non-trivial examples of the additional kinds that appear in the case with indefinite signature of the metric (Pucacco & Rosquist, 2009).

We plan to investigate the existence of higher-order polynomial integrals on general compact surfaces with the topology of the sphere and the torus and to apply the results about pseudo-Riemannian systems to treat integrable time-dependent Hamiltonian systems. We are also working on the general integrability conditions that a system must obey in order to be endowed with one or more integral of motion in a certain polynomial class.

3.3 Local dynamics in the expanding universe

The question of whether or not the expansion of the universe may have an influence on the dynamics of local systems such as galaxies has been a topic of debate almost since Einstein presented his theory of general relativity. In this project, we are exploiting a new approach to the problem in which the geometry of the universe appears as a perturbation of the flat metric at the position of the local system.

More specifically, to analyze how the universe influences the dynamics of a local system we take the position of an observer who is comoving with the universe. The local system is assumed to be a test system with respect to the universe. That is, the metric of the universe is to be regarded as a background for the local system. For an arbitrary local system, it has been very difficult to treat this problem exactly. However, there is one case which is known to be amenable to an exact treatment. This is the case of an electromagnetic system for which there is a well-known covariant formulation. The motion is then determined by the Lorentz force equation expressed on a curved background. This system was analyzed by Bonnor for a Bohr hydrogen type system. He came to the conclusion that a circular orbit expands during one revolution, but that the size of the effect is negligible compared to the Hubble expansion. According to our analysis, Bonnor's result should be modified. In fact, the effect turns out be the opposite, nameley that orbits are contracting rather than expanding. The physical reason behind this behavior is that the expansion of the universe leads to a loss of energy. The energy loss is formally analogous to a frictional dissipation. In technical terms, it is the absence of a timelike Killing symmetry which is responsible for the energy loss. Although the relation between energy conservation and time translation symmetry is well-known, that relation has previously received little attention in the cosmological context.

Using the electromagnetic interaction as a model, we have shown that is in fact possible to treat also other types of interaction in a similar way, in particular keeping full covariance. Our plan is to use this result to analyze in detail how orbits in different potentials are affected by the expansion of the universe. Of particular interest in this regard is to apply the method to realistic galactic potentials. Our preliminary work indicates that typical orbits will spiral inwards towards an inner limiting radius. In the coming year we intend to investigate what this behavior means for the galactic rotation curves.

4 Publications

1. Belmonte, C. Boccaletti, D. Pucacco, G. (2006) Stability of axial orbits in galactic potentials, Celestial Mechanics and Dynamical Astronomy, **95**, 101.

2. Belmonte, C. Boccaletti, D. Pucacco, G. (2007a) Approximate First Integrals for a Model of Galactic Potential with the Method of Lie Transform Normalization, Qualitative Theory of Dynamical Systems, E. Perez-Chavela and J. Xia editors.

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15. Pucacco, G. & Rosquist, K. (2007) (1+1)-dimensional separation of variables, J. Math. Phys., **48**, 112903.

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